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HOLOMORPHIC MAPPINGS OF POLYDISCS INTO COMPACT COMPLEX MANIFOLDS

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In this paper we prove an inequality in the manner of the Nevanlinna theory expressing certain properties of holomorphic mappings of n-dimensional polydiscs into compact complex manifolds of the same dimension and discuss some of its applications.

1. Let W be a compact complex manifold of dimension n. For a point w in W, we denote a local coordinate of w by $(w^1, w^2, \cdots, w^n)$. Take a complex line bundle L over W. By a theorem of de Rham, the Chern class $c(L)$ of L can be regarded as a d-cohomology *class* of d-closed 2-forms on W. We say that a real (1, 1)-form

$$\gamma = i \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(w) dw^\alpha \wedge d\bar{w}^\beta , \qquad i = \sqrt{-1} ,$$

on W is *positive semidefinite* (*or positive definite*) if the Hermitian matrix $(g_{\alpha\beta}(w))_{\alpha,\beta=1,\cdots,n}$ is positive semidefinite (or positive definite) at every point $w \in W$. Denote the canonical bundle of W by K. In this section we assume the existence of a complex line bundle L over W together with a positive integer m satisfying the following condition: *The Chern class $c(L)$ contains a positive semidefinite d-closed real* (1, 1)*-form and*

$$\dim H^0(W, \mathcal{O}(K^m \otimes L^{-1})) > 0 , \tag{1}$$

where $\mathcal{O}(K^m \otimes L^{-1})$ denotes the sheaf over W of germs of holomorphic sections of $K^m \otimes L^{-1}$.

Cover W by a *finite* number of small neighborhoods U_j, $j = 1, 2, \cdots$, and fix a local coordinate: $w \to (w_j^1, \cdots, w_j^n)$ on each U_j. Take a 1-cocycle $\{l_{jk}\}$ determining the line bundle L composed of nonvanishing holomorphic functions $l_{jk} = l_{jk}(w)$ defined, respectively, on $U_j \cap U_k$. We then find a 0-cochain $\{a_j\}$ composed of C^∞-differentiable functions $a_j = a_j(w) > 0$ defined, respectively, on U_j satisfying

$$a_j(w)^m = |l_{jk}(w)|^2 a_k(w)^m , \qquad \text{on } U_j \cap U_k ,$$

such that

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$$\gamma = i \sum_{\alpha,\beta=1}^{n} g_{j\alpha\beta}(w) dw_j^\alpha \wedge d\bar{w}_j^\beta = i\partial\bar{\partial} \log a_j(w)$$

is positive semidefinite. Note that the d-closed real (1, 1)-form $m\gamma$ belongs to the Chern class $c(L)$. We choose a holomorphic section

$$\varphi \in H^0(W, \mathcal{O}(K^m \otimes L^{-1})) , \qquad \varphi \neq 0 ,$$

and denote by $\varphi_j(w)$ the fibre coordinate of $\varphi(w)$ over U_j. It is clear that

$$v = a_j(w) | \varphi_j(w) |^{2/m} (i/2)^n dw_j^1 \wedge d\bar{w}_j^1 \wedge \cdots \wedge dw_j^n \wedge d\bar{w}_j^n$$

is a *volume element*, i.e., a real continuous $2n$-form which is nonnegative everywhere on W. Fix a point $p^0 \in W$ such that $\varphi(p^0) \neq 0$, and assume that $p^0 \in U_1$. We normalize the volume element v by the condition:

$$a_1(p^0) | \varphi_1(p^0) |^{2/m} = 1 . \tag{2}$$

Let $\mathbf{C}^n$ denote the space of n complex variables, define $|z| = \max_\lambda |z_\lambda|$ for $z = (z_1, \cdots, z_\lambda, \cdots, z_n) \in \mathbf{C}^n$, and denote by Δ_r a polydisc of radius r:

$$\Delta_r = \{ z \in \mathbf{C}^n \, | \, |z| < r \} .$$

Take a polydisc $\Delta_R \subseteq \mathbf{C}^n$, consider a holomorphic mapping f of Δ_R into W, and assume that the Jacobian of f does not vanish at the origin $0 \in \Delta_R$ and that

$$f(0) = p^0 . \tag{3}$$

For simplicity we write

$$dV(z) = (i/2)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n ,$$

and let $f^*(v)$ denote the volume element on Δ_R induced from v by the mapping f. Then we have

$$f^*(v) = \xi(z) dV(z) , \qquad \xi(z) = a_j(f(z)) | \varphi_j(f(z)) |^{2/m} | J_j(z) |^2 ,$$

where

$$J_j(z) = \det (\partial w_j^\alpha / \partial z_\lambda)_{\alpha, \lambda = 1, \cdots, n} , \qquad (w_j^1, \cdots, w_j^n) = f(z) .$$

By hypothesis the Jacobian $J_j(z)$ of f does not vanish identically, and therefore the equation $\xi(z) = 0$ defines a proper analytic subset of Δ_R. Hence, by applying a suitable linear transformation to $\mathbf{C}^n$ if necessary, we may assume that, for any fixed values of $z_1, \cdots, z_{\lambda-1}, z_{\lambda+1}, \cdots, z_n$, the function $\xi(z_1, \cdots, z_\lambda, \cdots, z_n)$ of z_λ does not vanish identically and that

$$J_1(0) = 1 . \tag{4}$$

Set

$$\sigma_\lambda = (i/2)^{n-1} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{\lambda-1} \wedge dz_{\lambda+1} \wedge \cdots \wedge d\bar{z}_n ,$$

$$\sigma = \sum_{\lambda=1}^{n} \sigma_\lambda ,$$

$$|\partial f(z)/\partial z_\lambda|^2 = \sum_{\alpha,\beta=1}^{n} g_{j\alpha\beta}(f(z))(\partial w_j^\alpha/\partial z_\lambda)(\partial \bar{w}_j^\beta/\partial \bar{z}_\lambda) ,$$

where $(w_j^1, \cdots, w_j^n) = f(z)$. Moreover, setting $z_\lambda = r_\lambda e^{i\theta_\lambda}$, we introduce polar coordinates $(r_\lambda, \theta_\lambda)$ and let

$$dS(z) = \sum_{\lambda=1}^{n} r_\lambda d\theta_\lambda \wedge \sigma_\lambda .$$

We denote the bundary of the polydisc Δ_r by $\partial\Delta_r$.

Now we define functions $M(r)$, $A(r)$ and $N(r)$ of r, $0 < r < R$, as follows:

$$M(r) = r^{-1} \int_{\partial\Delta_r} \log \xi(z) dS(z) ,$$

$$A(r) = 4 \int_{\Delta_r} \sum_{\lambda=1}^{n} |\partial f(z)/\partial z_\lambda|^2 dV(z) ,$$

$$N(r) = 4\pi m^{-1} \int_{(f^*\varphi)\cap\Delta_r} \sigma + 4\pi \int_{(J)\cap\Delta_r} \sigma ,$$

where $(f^*\varphi)$ and (J) denote, respectively, the divisors of the holomorphic functions $\varphi_j(f(z))$ and $J_j(z)$.

Theorem 1. *We have the inequality*:

$$\int_0^r A(t)t^{-1}dt + \int_0^r N(t)t^{-1}dt \leq M(r) . \tag{5}$$

Proof. Let

$$\mu(z) = \log \xi(z) .$$

The set $\Gamma = \{z \,|\, \xi(z) = 0\}$ is a proper analytic subset of Δ_R, and $\mu(z)$ is C^∞-differentiable outside Γ. For brevity we write

$$z = (z_1, \zeta) , \qquad \zeta = (z_2, \cdots, z_n) .$$

We set

$$\mu_1(r, \zeta) = \int_0^{2\pi} \mu(re^{i\theta}, \zeta) d\theta .$$

Lemma. *$\mu_1(r, \zeta)$ is a continuous function of (r, ζ), $0 < r < R$, $|\zeta| < R$, and is a piecewise smooth function of r, $0 < r < R$, when ζ is fixed.*

To prove this lemma, take a point ζ^0, $|\zeta^0| < R$, and a real number r^0, $0 < r^0 < R$, such that $(r^0 e^{i\theta}, \zeta^0) \notin \Gamma$ for $0 \leq \theta < 2\pi$. Moreover, for each $\zeta, |\zeta| < R$, denote by $\rho_h(\zeta), h = 1, 2, 3, \cdots$, the roots of the equation:

$$\varphi_j(f(z_1, \zeta)) J_j(z_1, \zeta)^m = 0 .$$

Then for a small positive number ε we have, for $|z_1| < r^0, |\zeta - \zeta^0| < \varepsilon$,

$$\mu(z) = 2m^{-1} \sum_h \log |z_1 - \rho_h(\zeta)| + \tau(z) ,$$

where the summation is extended over all roots $\rho_h(\zeta)$ with $|\rho_h(\zeta)| < r^0$, and $\tau(z)$ is a C^∞-differentiable function of z. Using the formula

$$\int_0^{2\pi} \log |re^{i\theta} - \rho| d\theta = 2\pi \max \{\log r, \log |\rho|\} ,$$

we hence obtain

$$\mu_1(r, \zeta) = 4\pi m^{-1} \sum_h \max \{\log r, \log |\rho_h(\zeta)|\} + \tau_1(r, \zeta) , \tag{6}$$

where $\tau_1(r, \zeta)$ is a C^∞-differentiable function of (r, ζ), $|r| < r_0$, $|\zeta - \zeta^0| < \varepsilon$. Since the roots $\rho_h(\zeta)$, arranged in an appropriate order, are continuous functions of $\zeta, |\zeta - \zeta^0| < \varepsilon$, the formula (6) proves the lemma.

Define

$$M(r_1, r_2, \cdots, r_n) = \int \mu(z_1, z_2, \cdots, z_n) d\theta_1 d\theta_2 \cdots d\theta_n ,$$

where the integral is extended over the domain: $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_2 < 2\pi$, $\cdots, 0 \leq \theta_n < 2\pi$. Since

$$M(r_1, r_2, \cdots, r_n) = \int \mu_1(r_1, z_2, \cdots, z_n) d\theta_2 \cdots d\theta_n ,$$

we infer from the above lemma that $M(r_1, r_2, \cdots, r_n)$ is a continuous function of $(r_1, r_2, \cdots, r_n) \neq (0, \cdots, 0)$, while, by (2), (3) and (4), the function $\mu(z)$ of z is C^∞-differentiable in a neighborhood of 0. Consequently $M(r_1, \cdots r_n)$ is a continuous function of $(r_1, \cdots, r_n)$, $0 \leq r_\lambda < R$.

Let ∂_1 denote the exterior differentiation with respect to the variable z_1. We then have

$$i\partial_1\bar{\partial}_1\mu(z) = i\partial_1\bar{\partial}_1 \log a_j(f(z)) = |\partial f(z)/\partial z_1|^2 i dz_1 \wedge d\bar{z}_1 .$$

Define

$$B(r, \zeta) = \int_{|z_1|<r} 2i\partial_1\bar{\partial}_1\mu(z) = \int_{|z_1|<r} 2|\partial f(z)/\partial z_1|^2 i dz_1 \wedge d\bar{z}_1 .$$

Setting $z_1 = x + iy$, we have

$$2i\partial_1\bar{\partial}_1\mu = d{*}d\mu, \qquad {*}d\mu = (\partial\mu/\partial x)dy - (\partial\mu/\partial y)dx .$$

Moreover the function $\mu(z_1, \zeta)$ is C^∞-differentiable in z_1 for $z_1 \neq \rho_h(\zeta)$. Hence, letting

$$\oint_\rho {*}d\mu(z) = \lim_{\varepsilon\to 0} \int_{|z_1-\rho|=\varepsilon} {*}d\mu(z_1, \zeta) ,$$

we obtain

$$B(r, \zeta) = \int_{|z_1|=r} {*}d\mu(z) - \sum_{|\rho|<r} \oint_\rho {*}d\mu(z) .$$

Note that $\oint_\rho {*}d\mu(z) = 0$ for $\rho \neq \rho_h(\zeta)$, $h = 1, 2, \cdots$. We denote by $\nu(r, \zeta, f^*\varphi)$ and $\nu(r, \zeta, J)$, respectively, the number of the roots on the disc $|z_1| < r$ of the equations $\varphi(f(z_1, \zeta)) = 0$ and $J_j(z, \zeta) = 0$. Since

$$\mu(z) = \log a_j(f(z)) + 2m^{-1} \log|\varphi_j(f(z))| + 2\log|J_j(z)| ,$$

we have

$$\sum_{|\rho|<r} \oint_\rho {*}d\mu(z) = 4\pi m^{-1}\nu(r, \zeta, f^*\varphi) + 4\pi\nu(r, \zeta, J) .$$

Moreover we see readily that

$$\int_{|z_1|=r} {*}d\mu(z) = r\partial\mu_1(r, \zeta)/\partial r .$$

Hence, setting

$$\nu(r, \zeta) = 4\pi m^{-1}\nu(r, \zeta, f^*\varphi) + 4\pi\nu(r, \zeta, J) ,$$

we obtain

$$B(r, \zeta) + \nu(r, \zeta) = r\partial\mu_1(r, \zeta)/\partial r ,$$

and therefore

$$(7) \qquad \int_s^r B(t,\zeta)t^{-1}dt + \int_s^r \nu(t,\zeta)t^{-1}dt = \mu_1(r,\zeta) - \mu_1(s,\zeta) .$$

This proves the inequality

$$\mu_1(r, z_2, \cdots, z_n) \geq \mu_1(s, z_2, \cdots, z_n) , \qquad \text{for } r > s > 0 .$$

It follows that

$$M(r, r_2, \cdots, r_n) \geq M(s, r_2, \cdots, r_n) , \qquad \text{for } r > s .$$

Thus we infer that $M(r_1, \cdots, r_\lambda, \cdots, r_n)$ *is a monotone nondecreasing function of each variable* r_λ. Since, by (2), (3) and (4), $\xi(0)$ is equal to 1, we get

$$(8) \qquad M(r_1, r_2, \cdots, r_n) \geq 0 .$$

Define

$$A(t,u) = \int_{|\zeta|\leq u} B(t,\zeta)dV(\zeta) ,$$

$$N(t,u) = \int_{|\zeta|\leq u} \nu(t,\zeta)dV(\zeta) ,$$

$$M_1(t,u) = \int_{|\zeta|\leq u} \mu_1(t,\zeta)dV(\zeta) ,$$

where

$$dV(\zeta) = \sigma_1 = (i/2)^{n-1}dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n .$$

Since $idz_\lambda \wedge d\bar{z}_\lambda = 2r_\lambda dr_\lambda d\theta_\lambda$, we have

$$M_1(r,u) = \int_0^u M(r, r_2, r_3, \cdots, r_n)r_2dr_2r_3dr_3\cdots r_ndr_n ,$$

where the integral is extended over the domain: $0 \leq r_\lambda \leq u, \lambda = 2, 3, \cdots, n$. Hence, using (8), we obtain from (7) the inequality

$$(9) \qquad \int_0^r A(t,u)t^{-1}dt + \int_0^r N(t,u)t^{-1}dt \leq M_1(r,u) .$$

Set

$$M_\lambda(r) = \int_0^r M(t_2, \cdots, t_\lambda, r, t_{\lambda+1}, \cdots, t_n)t_2dt_2\cdots t_ndt_n ,$$

$$A_\lambda(r) = 4\int_{\Delta_r} |\partial f(z)/\partial z_\lambda|^2 dV(z) ,$$

$$N_\lambda(r) = 4\pi m^{-1} \int_{(f^*\varphi)\cap\Delta_r} \sigma_\lambda + 4\pi \int_{(J)\cap\Delta_r} \sigma_\lambda .$$

Since $M_1(r) = M_1(r, r)$, $A_1(t) = A(t, t) \leq A(t, u)$ and $N_1(t) = N(t, t) \leq N(t, u)$ for $t \leq u$, we derive from (9) the inequality

$$\int_0^r A_1(t)t^{-1}dt + \int_0^r N_1(t)t^{-1}dt \leq M_1(t) .$$

We infer in the same manner that

$$\int_0^r A_\lambda(t)t^{-1}dt + \int_0^r N_\lambda(t)t^{-1}dt \leq M_\lambda(t) . \tag{10}$$

Since

$$rM(r) = \int_{\partial\Delta_r} \mu(z)dS(z) = \sum_{\lambda=1}^n \int_{|z|=|z_\lambda|=r} \mu(z)r_\lambda d\theta_\lambda \wedge d\sigma_\lambda ,$$

we have

$$M(r) = \sum_{\lambda=1}^n M_\lambda(r) ,$$

while it is obvious that

$$A(t) = \sum_{\lambda=1}^n A_\lambda(t) , \qquad N(t) = \sum_{\lambda=1}^n N_\lambda(t) .$$

Hence the inequality (5) follows from (10). q.e.d.

For a positive number β, we define

$$\Omega_\beta(r) = \int_{\partial\Delta_r} \xi(z)^\beta dS(z) ,$$

and set

$$S(r) = \int_{\partial\Delta_r} dS(z) = 2n\pi^n r^{2n-1} .$$

Theorem 2. *We have the inequality*

$$\int_0^r A(t)t^{-1}dt + \int_0^r N(t)t^{-1}dt \leq \beta^{-1}r^{-1}S(r)\log(\Omega_\beta(r)/S(r)) . \tag{11}$$

Proof. Since $\log x$ is a *concave* function of $x, x > 0$, we have

$$\begin{aligned} rM(r) &= \int_{\partial\Delta_r} \log \xi(z)dS(z) = \beta^{-1}\int_{\partial\Delta_r}\log \xi(z)^\beta dS(z) \\ &\leq \beta^{-1}S(r)\log\left(S(r)^{-1}\int_{\partial\Delta_r}\xi(z)^\beta dS(z)\right), \end{aligned}$$

which together with (5) gives the inequality (11). q.e.d.

We have assumed so far that the system of coordinates $(z_1, \cdots, z_\lambda, \cdots, z_n)$ is *general* in the sense that, for each λ and any fixed values of $z_1, \cdots, z_{\lambda-1}, z_{\lambda+1}, \cdots, z_n$, the function $\xi(z_1, \cdots, z_\lambda, \cdots, z_n)$ of z_λ does not vanish identically. However, this assumption is irrelevant to the inequality (11). *The inequality* (11) *holds for any system of coordinates* $(z_1, \cdots, z_n)$ *satisfying the conditions* (3) *and* (4). To prove this, suppose that the coordinates $(z_1, \cdots, z_n)$ are obtained from a fixed system of coordinates $(z_1^{(0)}, \cdots, z_n^{(0)})$ by means of a linear transformation $u = (u_{\lambda\nu})$ with $\det(u_{\lambda\nu}) = 1$:

$$z_\lambda = \sum_{\nu=1}^n u_{\lambda\nu}z_\nu^{(0)} .$$

There exists an everywhere dense subset G of the special linear group $SL(n, \mathbf{C})$ such that, for every $u \in G$, the corresponding system of coordinates $(z_1, \cdots, z_n)$ is general and, consequently, the inequality (11) holds. For our purpose it suffices, therefore, to verify that each term of (11) depends continuously on u. It is obvious that $\int_0^r A(t)t^{-1}dt$ and $\Omega_\beta(r)$ are continuous in u. Denoting the positive part of $\log x$ by $\log^+ x$, we have

$$\int_0^r N(t)t^{-1}dt = 4\pi m^{-1} \int_{(f^*\varphi)+m(J)} \log^+(r/|z|)\sigma ,$$

which shows that $\int_0^r N(t)t^{-1}dt$ depends continuously on u. q.e.d.

Note that

$$\int_{\Delta_r} \xi(z)^\beta dV(z) = \int_0^r \Omega_\beta(t)dt . \tag{12}$$

Since $A(t)$ and $N(t)$ are nonnegative, the inequality (11) implies that

(13) $$\Omega_\beta(r) \geq S(r) .$$

Combining this with (12), we get

(14) $$\int_{\Delta_r} \xi(z)^\beta dV(z) \geq \pi^n r^{2n} .$$

In particular, setting $\beta = 1$, we obtain

(15) $$\int_{\Delta_r} f^*(v) \geq \pi^n r^{2n} .$$

2. A holomorphic mapping is said to be *totally degenerate* if its Jacobian vanishes identically. Let v_0 be a volume element which is positive everywhere on W. Then, for any holomorphic mapping f of Δ_r into W, the quotient $\int_{\Delta_r} f^*(v_0) \Big/ \int_W v_0$ may be regarded as a *mean degree* of the mapping $f: \Delta_r \to W$. Define

$$\deg(f \,|\, \Delta_r) = \int_{\Delta_r} f^*(v_0) \Big/ \int_W v_0 ,$$

and further set

$$P_m = \dim H^0(W, \mathcal{O}(K^m)) , \qquad \text{for} \quad m = 1, 2, 3, \cdots .$$

Theorem 3. *Let W be a compact complex manifold of dimension n. If there exists a holomorphic mapping f of $\mathbf{C}^n$ into W which is not totally degenerate, and if*

(16) $$\liminf_{r \to +\infty} r^{-2n} \deg(f \,|\, \Delta_r) = 0 ,$$

then all the plurigenera P_m of W vanish.

Proof. Suppose that one of the plurigenera, say P_m, is positive. Then, letting L be a trivial bundle, we have the inequality (1). Hence, by (15), we obtain

$$\int_{\Delta_r} f^*(v) \geq \pi^n r^{2n} ,$$

which contradicts (16), since the quotient v/v_0 is bounded on W. q.e.d.

By a surface we shall mean a compact complex manifold of dimension 2. A surface W is said to be *regular* if the first Betti number $b_1(W)$ of W vanishes. *A regular surface W is rational if and only if all the plurigenera P_m of W vanish* (see [9, Theorem 54]).

Theorem 4. *If a regular surface W contains $\mathbf{C}^2$ as its open subset, then W is a rational surface.*

Proof. Let W be a regular surface containing $\mathbf{C}^2$ and let $f\colon \mathbf{C}^2 \subsetneq W$ denote the inclusion map. It is obvious that $\deg (f \mid \Delta_r) < 1$ for each polydisc $\Delta_r \subset \mathbf{C}^2$. Thus by Theorem 3 all the plurigenera P_m of W vanish, and hence W is a rational surface. q.e.d.

Letting U be a non-empty open subset of a compact complex manifold W, we call W a *compactification* of U if the complement $W - U$ of U is an analytic subset of W. F. Hirzebruch mentioned in his list [6] of problems the classification of all compactifications of $\mathbf{C}^n$. Concerning this problem, A. Van de Ven [13] pointed out that all the known examples of compactifications of $\mathbf{C}^2$ are rational surfaces.

Theorem 5. *Every compactification of $\mathbf{C}^2$ is a rational surface.*

Proof. Let W be a compactification of $\mathbf{C}^2$. It is then obvious that $b_1(W) = b_1(\mathbf{C}^2) = 0$. Hence, by Theorem 4, W is a rational surface. q.e.d.

The condition $\mathbf{C}^2 \subsetneq W$ is much weaker than that W is a compactification of $\mathbf{C}^2$. In fact, there exists an infinite sequence of *mutually disjoint* open subsets $U_1, U_2, U_3, \cdots$ of $\mathbf{C}^2$ each of which is biholomorphically isomorphic to $\mathbf{C}^2$ (see § 4 below). Thus, if $\mathbf{C}^2 \subsetneq W$, then $U_1 \subsetneq \mathbf{C}^2 \subsetneq W$, and the existence of $U_1 \subsetneq W$ together with the vanishing of $b_1(W)$ already implies the rationality of W.

3. Letting W be a projective algebraic manifold of dimension n, we call W an algebraic manifold of *general type* if

$$\limsup_{m \to +\infty} m^{-n} \dim H^0(W, \mathcal{O}(K^m)) > 0 \,, \tag{17}$$

where K denotes the canonical bundle of W. Recently Iitaka [7] introduced the concept of canonical dimension. The condition (17) is equivalent to saying that the canonical dimension of W coincides with the dimension n of W. In this section we apply Theorem 1 to algebraic manifolds of general type and derive a recent result of Griffiths [5].

Let W be an algebraic manifold of general type of dimension n, X a general hyperplane section of W, and $L = [X]$ the complex line bundle over W determined by the divisor X. Then, letting K_X denote the restriction of K to X, we have the exact sequence:

$$0 \to H^0(W, \mathcal{O}(K^m \otimes L^{-1})) \to H^0(W, \mathcal{O}(K^m)) \to H^0(X, \mathcal{O}(K_X^m)) \to \cdots \,,$$

while $\dim H^0(X, \mathcal{O}(K_X^m))$ is a function of m of order $O(m^{n-1})$. Hence, by (17), $\dim H^0(X, \mathcal{O}(K^m \otimes L^{-1}))$ is positive for a large integer m, and thus we have the inequality (1). Obviously we may assume that the real $(1, 1)$-form

$$i \sum g_{j\alpha\beta}(w) dw_j^\alpha \wedge d\bar{w}_j^\beta = i\partial\bar{\partial} \log a_j(w)$$

is *positive definite*. Therefore, setting

$$g_j(w) = \det\,(g_{j\alpha\beta}(w))\ ,$$

we find a positive constant c such that

$$a_j(w)\,|\,\varphi_j(w)\,|^{2/m} \leq c^n g_j(w)\ , \qquad \text{for } w \in W\ . \tag{18}$$

Now consider a holomorphic mapping $f\colon \varDelta_R \to W$ satisfying the conditions (3) and (4), and set

$$\Omega(r) = \Omega_{1/n}(r), \qquad T(r) = \int_{\varDelta_r} \xi(z)^{1/n} dV(z)\ .$$

Since

$$g_j(f(w))\,|\,J_j(z)\,|^2 \leq \prod_{\lambda=1}^{n} |\,\partial f(z)/\partial z_\lambda|^2\ ,$$

we have, in consequence of (18),

$$\xi(z) \leq c^n \prod_{\lambda=1}^{n} |\,\partial f(z)/\partial z_\lambda|^2\ , \qquad \xi(z)^{1/n} \leq n^{-1}c \sum_{\lambda=1}^{n} |\,\partial f(z)/\partial z_\lambda|^2\ ,$$

from which follows

$$T(r) \leq (4n)^{-1} c A(r)\ .$$

Combining this with (11) we obtain

$$\int_0^r T(t)t^{-1}dt \leq (4r)^{-1} c S(r) \log\,(\Omega(r)/S(r))\ . \tag{19}$$

Set

$$Q(r) = \int_0^r T(t)t^{-1}dt\ , \qquad \Psi(r) = 2n\pi^{-n}r^{-2n}Q(r)\ ,$$

and note that, by (14), $T(r) \geq \pi^n r^{2n}$, $Q(r) \geq (2n)^{-1}\pi^n r^{2n}$ and $\Psi(r) \geq 1$. The inequality (19) implies that

$$r \leq r_0\ , \qquad r_0 = r_0(c, n)\ ,$$

where $r_0(c, n)$ is a constant depending only on c and n (see Nevanlinna [11, p. 235]). In fact, if $\Omega(r) \leq r^2 Q(r)^4$, then the inequality (19) yields

$$r^2\Psi(r) \leq n^2 c(4 \log \Psi(r) + (6n + 3) \log r + 3n \log \pi)\ .$$

Since $\Psi(r) \geq 1$ and $e \log x \leq x$ for $x > 0$, this proves that

$$r \leq r_1 = \max\{1, n^2 c e^{-1}(6n + 7) + 3n \log \pi\} .$$

Therefore, if $r > r_1$, then (19) implies that $\Omega(r) > r^2 Q(r)^4$. It follows that either $\Omega(r) > T(r)^2$ or $T(r) > rQ(r)^2$. If $\Omega(r) > T(r)^2$, then

$$dr = \Omega(r)^{-1} dT(r) < T(r)^{-2} dT(r) .$$

If $T(r) > rQ(r)^2$, then

$$dr = T(r)^{-1} r dQ(r) < Q(r)^{-2} dQ(r) .$$

Hence we get

$$\begin{aligned} r - r_1 = \int_{r_1}^{r} dt &< - \int_{r_1}^{r} d(T(t)^{-1} + Q(t)^{-1}) \\ &< T(r_1)^{-1} + Q(r_1)^{-1} < (2n + 1)\pi^{-n} , \end{aligned}$$

which proves that

$$r \leq r_0 , \qquad r_0 = r_1 + (2n + 1)\pi^{-n} .$$

Thus we obtain the following

Theorem 6. *Let W be an algebraic manifold of general type, and p^0 a point on W such that $\varphi(p^0) \neq 0$ for an element $\varphi \in H^0(W, \mathcal{O}(K^m \otimes L^{-1}))$. Then there exists a constant r_0 with the following properties: For any holomorphic mapping $f\colon \Delta_R \to W$ with $f(0) = p^0$ and $J_1(0) = 1$, the inequality $R \leq r_0$ holds, where $J_1(0)$ denotes the Jacobian of f at the origin* 0.

This theorem has been proved by Griffiths [5] under the assumption that the canonical system $|K|$ is ample. We remark that his proof also applies to the case in which $|K|$ is not assumed to be ample, and establishes the above Theorem 6 (see Kobayashi and Ochiai [8, Addendum]).

4. Bieberbach [2] constructed an example of a biholomorphic mapping f of $\mathbf{C}^2$ onto a proper open subset U of $\mathbf{C}^2$. His construction is as follows. Let $\eta\colon z \to \eta z$ be a biholomorphic automorphism of $\mathbf{C}^2$ of which the origin 0 is a fixed point: $\eta 0 = 0$. Obviously η induces a linear transformation of the tangent space $T_0(\mathbf{C}^2)(\cong \mathbf{C}^2)$ of $\mathbf{C}^2$ at 0. Let λ and μ denote the eigenvalues of this linear transformation, and assume that $|\lambda| \leq |\mu| < 1$. Then there exists a biholomorphic mapping $f_0\colon z \to f_0(z)$ of a neighborhood N of 0 into $\mathbf{C}^2$ with $f_0(0) = 0$ such that $g = f_0^{-1} \eta f_0$ takes the normal form

$$g\colon z = (z_1, z_2) \to gz = (\lambda z_1 + \beta z_2^p, \mu z_2) ,$$

where p is a positive integer and β is a constant which vanishes unless $\lambda = \mu^p$ (see Lattès [10], Sternberg [12]). Obviously g is a contraction in the sense that

$$\lim_{m \to +\infty} g^m z = 0 , \qquad \text{for } z \in \mathbf{C}^2 .$$

For every positive integer m, we have

$$f_0(z) = \eta^{-m} f_0(g^m z) , \qquad \text{for } z \in N , \tag{20}$$

provided that $gN \subset N$. Since $\eta^{-m} f_0 g^m$ is defined on $g^{-m}N$ and $\bigcup_m g^{-m}N = \mathbf{C}^2$, it follows from (20) that f_0 can be continued analytically to a biholomorphic mapping f of $\mathbf{C}^2$ onto an open subset U of $\mathbf{C}^2$ (see Sternberg [12, p. 816]). For every integer m we have

$$f(z) = \eta^{-m} f(g^m z) , \qquad \text{for } z \in \mathbf{C}^2 .$$

It follows that

$$U = \{z \,|\, \lim_{m \to +\infty} \eta^m z = 0\} .$$

Now we specify η to be the automorphism

$$\eta \colon z = (z_1, z_2) \to \eta z = (z_2, \lambda^2 z_1 + (\lambda^2 - 1)(\sin z_2 - z_2)) ,$$

where λ is a constant with $0 < |\lambda| < 1$. Note that the normal form of this η is

$$g \colon z = (z_1, z_2) \to gz = (\lambda z_1, -\lambda z_2) .$$

We define a translation

$$\tau \colon z = (z_1, z_2) \to (z_1 + 2\pi, z_2 + 2\pi) .$$

Then η and τ are commutative: $\eta\tau = \tau\eta$, and therefore, for each integer k, $\tau^k 0 = (2k\pi, 2k\pi)$ is a fixed point of η and

$$\tau^k U = \{z \,|\, \lim_{m \to +\infty} \eta^m z = \tau^k 0\} .$$

It follows that $\tau^k U$ and $\tau^j U$ are disjoint for $k \neq j$. Thus we obtain an infinite sequence of mutually disjoint open subsets $\tau^k U, k = 0, \pm 1, \pm 2, \cdots$, each of which is biholomorphically isomorphic to $\mathbf{C}^2$.

Letting $\{\tau\}$ denote the infinite cyclic group generated by τ, we have

$$\mathbf{C}^2 / \{\tau\} = \mathbf{C}^* \times \mathbf{C} .$$

Clearly we may regard $U = \bigcup_k \tau^k U / \{\tau\}$ as an open subset of $\mathbf{C}^* \times \mathbf{C}$. Thus we see the existenc of a biholomorphic mapping: $\mathbf{C}^2 \hookrightarrow \mathbf{C}^* \times \mathbf{C}$. Combining this with Theorem 4, we infer that if a regular surface W contains $\mathbf{C}^* \times \mathbf{C}$ as its open subset, then W is a rational surface. This result can be verified also in the same manner as in the proof of Theorem 4. In fact, if $\mathbf{C}^* \times \mathbf{C} \subset W$, then

$f: (z_1, z_2) \rightarrow (\exp z_1, z_2)$ is a holomorphic mapping of $\mathbf{C}^2$ into W with $\deg (f | \Delta_r) = O(r)$. Thus by Theorem 3 all the plurigenera of W vanish, and hence W is a rational surface.

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